**Self-assessment answers: 25 Mathematical induction**

**1.** Proposition: 

For *n* = 1:

LHS = 1 × 3 = 3; RHS =  = 3

Assume the proposition is true for *n* = *k*,



Let *n* = *k* + 1. Working towards:



LHS =  using the assumption.





 = RHS

So the result is true for *n* = 1 and if true for *n* = *k* then it is true for *n* = *k* + 1. Therefore, by the principle of mathematical induction, it is true for all *n* ∈ ℤ+.

*[6 marks]*

**2.** Proposition: *f* (*n*) = 15*n* – 2*n* is divisible by 13.

*f* (0) = 1 – 1 = 0 = 0 × 13, so *f* (0) is divisible by 13.

Assume that *f* (*k*) is divisible by 13. Then *f* (*k*) = 13*A* for some *A* ∈ ℤ.

When *n* = *k* + 1, we are working towards *f* (*k* + 1) = 15*k* + 1 – 2*k* + 1 = 13*B* for some B ∈ ℤ.

LHS = 15 × (15*k* – 2*k*) + 13 × 2*k* + 1 using the assumption.

= 15 × 13*A* + 13 × 2k + 1

= 13(15*A* + 2*k* + 1) = RHS with *B* = 15*A* + 2*k* + 1

So *f* (*k* + 1) is divisible by 13.

*f* (0) is divisible by 13 and if *f* (*k*) is divisible by 13 then so is *f* (*k* + 1).Therefore, by the principle of mathematical induction, *f* (*n*) is divisible by 13 for all n ∈ ℕ. *[6 marks]*

**3.** Proposition: *f* (*n*) = 3*n* > *n* + 17 for *n* ≥ 3.

*f* (3) = 27 > 20 = 3 + 17, so the proposition is true for *n* = 3.

Assume that *f* (*k*) > *k* + 17.

When *n* = *k* + 1, we are working towards *f* (*k* + 1) > *k* + 18.

LHS = 3 × *f* (*k*) > 3 × (*k* + 17), using the assumption

LHS > *k* + 18 + (2*k* + 33) > *k* + 18 since 2*k* + 33 > 0 for *k* ≥ 3.

LHS > RHS, so the proposition holds for *n* = *k* + 1.

The inequality holds for *n* = 3 and if it holds for *n* = *k* then it also holds for *n* = *k* + 1. Therefore, by the principle of mathematical induction, it holds for all integers *n* ≥ 3. *[6 marks]*

**4.** (a) Proposition: (cos *θ* + i sin *θ*)*n* = cos(*nθ*) + i sin(*nθ*).

For *n* = 0:

LHS = 1; RHS = cos(0) + i sin(0) = 1 + 0i = 1.

Assume the proposition is true for n = *k*: (cos *θ* + i sin *θ*)*k* = cos(*kθ*) + i sin(*kθ*).

Let *n* = *k* + 1.

Working towards: (cos *θ* + i sin *θ*)*k + 1* = cos((*k* + 1)*θ*) + i sin((*k* + 1)*θ*).

LHS = (cos *θ* + i sin *θ*)[cos(*kθ*) + i sin(*kθ*)]

= cos *θ* cos(*kθ*) – sin *θ* sin(*kθ*) + i(cos *θ* sin(*kθ*) + sin *θ* cos(*kθ*))

= cos((*k* + 1)*θ*) + i sin((*k* + 1)*θ*) = RHS, by compound angle formulae.

So the result is true for *n* = 0 and if true for *n* = *k* then it is true for *n* = *k* + 1. Therefore, by the principle of mathematical induction, it is true for all *n* ∈ ℕ.

(b) Using (a), (cos *θ* + i sin *θ*)5 = cos(5*θ*) + i sin(5*θ*).

Hence, sin(5*θ*) = Im[(cos *θ* + i sin *θ*)5]

= Im[cos5 *θ* + 5i cos4 *θ* sin *θ* – 10 cos3 *θ* sin2 *θ* – 10i cos2 *θ* sin3 *θ* + 5 cos *θ* sin4 *θ* + i sin5 *θ*]

= 5 cos4 *θ* sin *θ* – 10 cos2 *θ* sin3 *θ* + sin5 *θ*

= 5(1 – sin2 *θ*)2sin *θ* – 10(1 – sin2 *θ*)sin3 *θ* + sin5 *θ*

5 sin *θ* – 20 sin3 *θ* + 16 sin5 *θ [12 marks]*